

# Global wellposedness for a certain class of large initial data for the 3D Navier-Stokes Equations

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## Abstract

In this article, we consider a special class of initial data to the 3D Navier-Stokes equations on the torus, in which there is a certain degree of orthogonality in the components of the initial data. We showed that, under such conditions, the Navier-Stokes equations are globally wellposed. We also showed that there exists large initial data, in the sense of the critical norm  $B_{\infty,\infty}^{-1}$  that satisfies the conditions that we considered.

## 1 Introduction and Summary

In this paper, we study the global wellposedness problem on the 3-torus with side length 1,  $\mathbb{T}^3$ , for the Navier-Stokes system:

$$\begin{cases} u_t + u \cdot \nabla u - \Delta u + \nabla p = 0 & \text{in } \mathbb{T}^3 \times \mathbb{R}_+ \\ \operatorname{div} u = 0 & \text{in } \mathbb{T}^3 \times \mathbb{R}_+ \\ u(x, 0) = u_0(x) & \text{on } \mathbb{T}^3 \times \{t = 0\} \end{cases} \quad (1)$$

with the restriction on  $u_0 = (u_0^1, u_0^2, u_0^3)$  that the frequency support of the three components are disjoint, and that they are sufficiently separated in the frequency space.

The goal of this paper is to answer in the affirmative the global wellposedness problem above:

**Theorem 1.** *Suppose  $u_0^i$  are a mean zero functions with disjoint frequency support, i.e.  $\text{supp}(\widehat{u}_i) \cap \text{supp}(\widehat{u}_j) = \emptyset$  for  $i \neq j$ . Suppose furthermore that, two of the three components, say  $u_0^1$  and  $u_0^2$  are frequency localized at scale  $N$  and  $N^{1+\epsilon}$  respectively for some  $\epsilon > 0$ , i.e.  $|\text{supp}(\widehat{u}_i)| \subset [N, O(1)N]$  and  $|\text{supp}(\widehat{u}_i)| \subset [N^{1+\epsilon}, O(1)N^{1+\epsilon}]$ . Then there exists  $\delta(\epsilon) > 0$  and some universal constant  $C$  such that if  $\|u_0^i\|_{L^2} \leq (C^{-1}\delta \log N^{1+\epsilon})^{1/2}$  for  $1 \leq i \leq 3$ , and  $\|u_0^3\|_{L^3} \leq C^{-1}N^{\frac{1}{3}-\delta(1+\epsilon)}(\log N^{1+\epsilon})^{-1/2}$ , the problem (1) admits a unique global solution in  $C_b(\mathbb{R}_+, L^3)$ .*

**Remark 1.** As we shall see in the proof, the sole purpose of the parameter  $\epsilon$  in the statement of the theorem is to allow us largeness in all three components of the initial data. If one is only interested in a global existence result that allows two of the three components to be large and the remaining one small in  $B_{\infty,\infty}^{-1}$  (but still large in the lower critical Besov norms), one can remove the  $\epsilon$  in the statement and proceed as in the proof below easily. However, as of now, there is no satisfactory small data wellposedness results for the 3D Navier-Stokes in  $B_{\infty,\infty}^{-1}$  and given the work of Bourgain and Pavlović [2], it is unlikely that one can be obtained through a classical perturbative argument. As a result, even if one of the components remain small in  $B_{\infty,\infty}^{-1}$ , the data will not fall within the established framework of a small perturbation of a 2D data.

**Remark 2.** The reason for the problem to be posed on the torus instead of  $\mathbb{R}^3$  is to avoid having initial data with infinite energy. If one is allowed that, the result can be extended to  $\mathbb{R}^3$ .

We shall also show that the conditions set out in the theorem above admits arbitrarily large data in  $B_{\infty,\infty}^{-1}$  that is not a perturbation of a 2D data:

**Theorem 2.** *For any  $M > 0$ , there exists initial data  $u_0^1, u_0^2, u_0^3$  satisfying that conditions in Theorem 1 such that each component  $u_0^i$  is large in the following sense:*

$$\|u_0^i\|_{B_{\infty,\infty}^{-1}} > M$$

The study of wellposedness of the Navier-Stokes equations is one with a long history, originated by the seminal paper of Leray [17], proving the existence of weak solutions. The study of the equations for small data in critical spaces started with a groundbreaking paper by Fujita and Kato [14], in which they showed local wellposedness and small data global wellposedness in the space  $H^{\frac{1}{2}}$ . Later, Kato [15] also showed the same results in the larger critical space  $L^3$ . In the early nineties, Cannone, Meyer and Planchon [4] showed small data global wellposedness in the spaces  $B_{p,\infty}^{-1+\frac{3}{p}}$  and more recently Koch and Tataru [16] showed the same result in the spaces of derivatives of  $BMO$  functions  $\partial BMO$ .

The largest critical space known is the space  $B_{\infty,\infty}^{-1}$ , for which Bourgain and Pavlović [2] have provided an example with arbitrarily small norm that blows up in arbitrarily short amount of time.

To summarize we have the following embeddings of critical spaces:

$$H^{1/2} \hookrightarrow L^3 \hookrightarrow B_{p,\infty}^{-1+\frac{3}{p}} \hookrightarrow \partial BMO \hookrightarrow B_{\infty,\infty}^{-1}$$

where small data global wellposedness have been proven for all except the last space, in which illposedness (in the sense that arbitrarily small data can grow to arbitrarily large magnitude in arbitrarily small time) exists.

The study for large initial data problems, where the large initial data does not reduce to a perturbation of the 2D problems or a formulation where exact solutions or global wellposedness is already known, started with a recent paper by Chemin and Gallagher [6], and continued by Bahouri, Chemin, Gallagher, Mullaert, Paicu and Zhang (e.g. [7], [8], [10], [11], [1], [9]).

## 2 Preliminary considerations

Before we begin proving the theorems, let us first make some simple observations that are going to greatly reduce the complexity of the problem at hand. First of all, the incompressibility condition in frequency space reads:

$$\sum_{i=1}^3 \xi_i \hat{u}^i(\xi) = 0$$

Under the conditions that initially the frequency supports of the three components of the data are disjoint, we can infer that for  $1 \leq i \leq 3$ ,

$$\xi_i \hat{u}^i(\xi) = 0$$

Interpreting the above in physical space, we have, initially the data  $u_0^1$  is a function of  $x_2$  and  $x_3$  only,  $u_0^2$  a function of  $x_1$  and  $x_3$  only and  $u_0^3$  a function of  $x_1$  and  $x_2$  only. We shall therefore make use of this observation and forget about the disjointness of the frequency support from now on.

Let us from now on denote the Leray projector onto the divergence free vector field by  $\mathbf{P}$ . The problem (1) can be simplified by breaking it apart into four problems. The first three are two-dimensional Navier-Stokes equations with each component as its initial condition. For example the first component  $u_0^1$  would yield:

$$\begin{cases} v_t^1 + \mathbf{P}(v^1 \cdot \nabla v^1) - \Delta v^1 = 0 & \text{in } \mathbb{T}^3 \times \mathbb{R}_+ \\ \operatorname{div} v^1 = 0 & \text{in } \mathbb{T}^3 \times \mathbb{R}_+ \\ v^1(x, 0) = (u_0^1(x), 0, 0)^t & \text{on } \mathbb{T}^3 \times \{t = 0\} \end{cases} \quad (2)$$

Note that this is possible because  $(u_0^1, 0, 0)$  satisfies the incompressibility condition and similarly for the other two components. Upon further inspection, it is clear that the nonlinear term  $\mathbf{P}(u \cdot \nabla u)$  vanishes for all time as  $\frac{\partial u_0^1}{\partial x_1} = 0$  for all  $x$ . Thus, the problem (2) is just a heat equation and solution is  $v^1(x, t) = (e^{t\Delta} u_0^1, 0, 0)^t$ . To make the observation above rigorous, one only needs to invoke the uniqueness of solutions to the 2D Navier-Stokes equations and observe that the solution to the 2D heat equation also solves the 2D Navier-Stokes for our prescribed initial conditions.

The fourth equation, which will be the main focus of the rest of this paper, accounts for the interaction of these solutions to heat equations:

$$\begin{cases} R_t + \mathbf{P}(R \cdot \nabla R) - \Delta R + Q(v^*, R) = F & \text{in } \mathbb{T}^3 \times \mathbb{R}_+ \\ \operatorname{div} R = 0 & \text{in } \mathbb{T}^3 \times \mathbb{R}_+ \\ R(x, 0) = 0 & \text{on } \mathbb{T}^3 \times \{t = 0\} \end{cases} \quad (3)$$

where, borrowing the notation from [6],  $Q(a, b) = \mathbf{P}\operatorname{div}(a \otimes b + b \otimes a)$ ,  $F = -\sum_{i < j} Q(v^i, v^j)$ ,  $v^i$  the solution to the  $i$ -th heat equation as above, and  $v^* = \sum_i v^i$ .

The equation above is the usual incompressible Navier-Stokes with a linear term and an inhomogenous forcing term. For small data, Chemin and Gallagher [6] has shown that the system is globally wellposed given good decay properties of  $v^*$ . The reason is firstly the system retains all the scaling properties of Navier-Stokes; and secondly under small data, the system is govern by the linear term, and the decay properties of  $v^*$  will guarantee that the evolution of the equation remains small in the suitable norms. This formulation thus suggests that, if we are able to control the behaviour of  $F$ , that is if we can control the interaction of the solutions of heat solutions which propagates in orthogonal directions, then a perturbative argument will yield us global wellposedness of the original Navier-Stokes problem.

Lastly, as we see that a good understanding of the decay property of the heat kernel is required to solve the problem at hand, we state without proof the following lemma for the readers' convenience. The lemma can be proven easily by analysing the fundamental solution of the heat equation.

**Lemma 1.** *We have the following  $L^p$  to  $L^q$  estimates for the 3-dimensional heat equation:*

$$\|\partial^s e^{t\Delta} u\|_{L^q} \leq t^{-\frac{3+s}{2} + \frac{3}{2}(\frac{1}{p} - \frac{1}{q} + 1)} \|u\|_{L^p}$$

### 3 Notations

In this paper, we will be working in  $L^p$  spaces, the Sobolev spaces  $W^{1,p}$ , as well as various Besov spaces  $B_{p,q}^s$ . Let us take a moment to establish some simple definitions and facts about these spaces.

**Definition 1.** Let  $P_j$  denote the frequency cut-off operator of the  $j$ -th dyadic frequencies,

$$\widehat{P_j f}(\xi) = \chi(\xi/2^j) \hat{f}(\xi)$$

where  $\chi(\xi)$  is a function supported on  $1/2 \leq |\xi| \leq 2$  and for all  $\xi \neq 0$ ,

$$\sum_{j \in \mathbb{Z}} \chi(\xi/2^j) = 1$$

We say  $f$  belongs to the Besov space  $B_{p,q}^s(\mathbb{T}^3)$ ,  $s \in \mathbb{R}$ ,  $1 \leq p, q \leq \infty$ , if  $f$  is a distribution such that

$$\|f\|_{B_{p,q}^s} := \|2^{js} \|P_j f\|_{L^p}\|_{l_j^q(\mathbb{Z})} < \infty$$

An equivalent definition of the Besov spaces on  $\mathbb{T}^3$  and for  $s$  negative is

$$\|f\|_{B_{p,q}^s} := \|t^{-s/2} \|e^{t\Delta} f\|_{L^p}\|_{L^q(\mathbb{R}_+, \frac{dt}{t})} < \infty$$

The proof for the equivalence of definitions and many other useful properties of Littlewood-Paley decompositions and Besov spaces can be found in [12]. For the purpose of this paper, we will only require the following embeddings:

**Lemma 2.** *In  $\mathbb{T}^3$ , we have the inclusion of the following spaces:*

$$L^3 \hookrightarrow B_{p,2}^{-1+\frac{3}{p}} (p > 3) \hookrightarrow B_{\infty,\infty}^{-1}$$

*In  $\mathbb{T}^2$ , we have the following inclusions:*

$$L^2 \hookrightarrow B_{\infty,2}^{-1}$$

The proof of the above lemma is a straightforward application of the Bernstein's inequality and will be omitted.

## 4 Estimates for $F$

We start by estimating the  $L_t^1 L_x^3$  norm of  $F$ . Given theorem 3 and lemma 2, this is indeed the correct space to consider. The terms in  $F$  is of the form  $e^{t\Delta} u^i \partial_i e^{t\Delta} u^j$  where  $i \neq j$ . By the triangle inequality, it is enough to establish a bound on the  $L_t^1 L_x^3$  norm of  $e^{t\Delta} u^i \partial_i e^{t\Delta} u^j$ ,  $i \neq j$ . We will consider the case where  $u^i$  concentrates around frequency  $N$  and  $u^j$  concentrates around frequency  $N^{1+\epsilon}$  and the case with  $u^i$  concentrates around  $N^{1+\epsilon}$  and  $u^j$  concentrates around  $N$ . The remaining cases where at least one of  $u^i$  or  $u^j$  is of low frequency can be estimated similarly and shall be omitted. Without loss of generality, we shall assume  $i = 1$  and  $j = 2$ .

**Case 1**  $|\text{supp}(\widehat{u^1})| \subset [N, O(1)N]$ ,  $|\text{supp}(\widehat{u^2})| \subset [N^{1+\epsilon}, O(1)N^{1+\epsilon}]$

$$\begin{aligned}
& \int_0^\infty \|e^{t\Delta} u^1 \partial_1 e^{t\Delta} u^2\|_{L^3} dt \\
& \leq \int_0^\infty \|e^{t\Delta} u^1\|_{L_{x_2}^3 L_{x_3}^3} \|\partial_1 e^{t\Delta} u^2\|_{L_{x_1}^3 L_{x_3}^\infty} dt \\
& \lesssim \int_0^K \|u^1\|_{L^3} t^{-1/2} N^{\frac{1+\epsilon}{3}} \|u^2\|_{L^3} dt + \int_K^\infty \|u^1\|_{L^3} \|\partial^{4/3} e^{t\Delta} u^2\|_{L^3} dt \\
& \lesssim \|u^1\|_{L^3} \|u^2\|_{L^3} K^{1/2} N^{\frac{1+\epsilon}{3}} + \int_K^\infty \|u^1\|_{L^3} \|\partial^4 e^{t\Delta} \partial^{-8/3} u^2\|_{L^3} dt \\
& \lesssim \|u^1\|_{L^3} \|u^2\|_{L^3} K^{1/2} N^{\frac{1+\epsilon}{3}} + \int_K^\infty \|u^1\|_{L^3} \|u^2\|_{L^3} t^{-2} N^{\frac{-8(1+\epsilon)}{3}} dt \\
& \lesssim \|u^1\|_{L^3} \|u^2\|_{L^3} (K^{1/2} N^{\frac{1+\epsilon}{3}} + K^{-1} N^{\frac{-8(1+\epsilon)}{3}})
\end{aligned}$$

Here, we use Holder inequality in the second line, Bernstein's inequality and fact that the heat flow does not alter the frequency of the data in the first term of the third line, and Sobolev inequality in the second term of the third line. Bernstein's inequality is used once again in the second term of the fifth line.

The quantity inside the bracket is minimized at the natural scaling  $K = N^{-2(1+\epsilon)}$ , substituting in, we have the estimate:

$$\int_0^\infty \|e^{t\Delta} u^1 \partial_1 e^{t\Delta} u^2\|_{L^3} dt \lesssim \|u^1\|_{L^3} \|u^2\|_{L^3} N^{-2/3-2\epsilon/3}.$$

**Case 2**  $|\text{supp}(\widehat{u^1})| \subset [N^{1+\epsilon}, O(1)N^{1+\epsilon}]$ ,  $|\text{supp}(\widehat{u^2})| \subset [N, O(1)N]$

$$\begin{aligned}
& \int_0^\infty \|e^{t\Delta} u^1 \partial_1 e^{t\Delta} u^2\|_{L^3} dt \\
& \leq \int_0^\infty \|e^{t\Delta} u^1\|_{L^3_{x_2} L^\infty_{x_3}} \|\partial_1 e^{t\Delta} u^2\|_{L^3} dt \\
& \lesssim \int_0^K N^{(1+\epsilon)/3} \|u^1\|_{L^3} t^{-1/2} \|u^2\|_{L^3} dt + \int_K^\infty \|\partial^{1/3} e^{t\Delta} u^1\|_{L^3} t^{-1/2} \|u^2\|_{L^3} dt \\
& \lesssim \|u^1\|_{L^3} \|u^2\|_{L^3} N^{(1+\epsilon)/3} K^{1/2} + \int_K^\infty \|\partial^3 e^{t\Delta} \partial^{-8/3} u^1\|_{L^3} t^{-1/2} \|u^2\|_{L^3} dt \\
& \lesssim \|u^1\|_{L^3} \|u^2\|_{L^3} (N^{(1+\epsilon)/3} K^{1/2} + \int_K^\infty t^{-2} N^{-8(1+\epsilon)/3} dt) \\
& \lesssim \|u^1\|_{L^3} \|u^2\|_{L^3} (N^{(1+\epsilon)/3} K^{1/2} + K^{-1} N^{-8(1+\epsilon)/3})
\end{aligned}$$

From here we proceed as in case 1 to obtain the same conclusion. As mentioned above, the other estimates where one of the term is of low frequency can be obtained using similar methods.

To summarize, we have proven the following:

**Proposition 1.** *The following estimates hold for the inhomogenous term  $F$  for equation (3)*

$$\|F\|_{L_t^1 L_x^3} \leq C_1 \|u^i\|_{L^3} \|u^j\|_{L^3} N^{-2/3-2\epsilon/3} \quad (4)$$

if one of  $u^i$  or  $u^j$  has frequency support at scale  $N^{1+\epsilon}$ . Otherwise, we have

$$\|F\|_{L_t^1 L_x^3} \leq C_1 \|u^i\|_{L^3} \|u^j\|_{L^3} N^{-2/3} \quad (5)$$

## 5 Global wellposedness for the residual system

In this section, we would like to prove global wellposedness for the residual system (3), thus completing the proof of the main theorem:

**Proposition 2.** *Assuming the conditions for the initial data listed in theorem (1), the system (3) has a unique global solution in  $C_b(\mathbb{R}_+; L^3)$*

The proof of it makes use of the following theorem and proposition proven in [6]:

**Theorem 3.** *Let  $p \in (3, \infty)$  be given. There is a constant  $C_0 > 0$  such that for any  $R_0$  in  $B_{p,2}^{-1+\frac{3}{p}}$ ,  $F$  in  $L^1(\mathbb{R}_+; B_{p,2}^{-1+\frac{3}{p}})$  and  $v^*$  in  $L^2(\mathbb{R}_+; L^\infty)$  satisfying*

$$\|R_0\|_{B_{p,2}^{-1+\frac{3}{p}}} + \|F\|_{L^1(\mathbb{R}_+; B_{p,2}^{-1+\frac{3}{p}})} \leq C_0^{-1} e^{-C_0 \|v^*\|_{L^2(\mathbb{R}_+; L^\infty)}^2} \quad (6)$$

*there is a unique global solution  $R$  to (3) associated with  $R_0$  and  $F$ , such that*

$$R \in C_b(\mathbb{R}_+; H^{1/2}) \cap L^2(\mathbb{R}_+; H^{3/2})$$

*Proof.* It is easy to check that the assumptions on the initial data are enough to satisfy (6). Indeed we have, firstly,

$$\|v^*\|_{L^2(\mathbb{R}_+; L^\infty)}^2 \leq \sum_{1 \leq i \leq 3} \|u_0^i\|_{B_{\infty,2}^{-1}}^2$$

by definition. From the embedding of  $L^2$  into  $B_{\infty,2}^{-1}$ , we have

$$\|u_0^i\|_{B_{\infty,2}^{-1}} \lesssim \|u_0^i\|_{L^2}$$

which in turn is bounded by  $(C_0^{-1} \delta \log N^{1+\epsilon})^{1/2}$  by our assumptions. Therefore we have

$$e^{-C_0 \|v^*\|_{L^2(\mathbb{R}_+; L^\infty)}^2} \geq N^{-\delta(1+\epsilon)}$$

On the other hand, since  $L^3(\mathbb{T}^3) \hookrightarrow B_{p,2}^{-1+\frac{3}{p}}(\mathbb{T}^3)$  for  $p > 3$ , we have

$$\|F\|_{L^1(\mathbb{R}_+; B_{p,2}^{-1+\frac{3}{p}})} \leq C_2 \|F\|_{L^1(\mathbb{R}_+; L^3)}$$

From the assumption of the energy of the initial condition, and by Bernstein's inequality,  $\|u_0^1\|_{L^3} \lesssim N^{\frac{1}{3}} (C^{-1} \delta \log N^{1+\epsilon})^{1/2}$  and  $\|u_0^2\| \lesssim N^{\frac{1+\epsilon}{3}} (C^{-1} \delta \log N^{1+\epsilon})^{1/2}$ . Together with the assumption on the  $L^3$  norm of  $\|u_0^3\|$ , we have the following:

For  $i \neq 2$ ,

$$\|u_0^i\|_{L^3} \|u_0^2\|_{L^3} \lesssim N^{\frac{2+\epsilon}{3}} C^{-1} \delta \log N^{1+\epsilon}$$

and

$$\|u_0^1\|_{L^3} \|u_0^3\|_{L^3} \lesssim (C^{-1} \delta)^{1/2} N^{\frac{2}{3}-\delta(1+\epsilon)}$$

From the estimates (4) and (5), and for sufficiently small  $\delta$ , say less than  $\epsilon/4$ , we can conclude that

$$\|F\|_{L^1(\mathbb{R}_+; L^3)} \leq C^{-1} N^{-\delta(1+\epsilon)}$$

□



Condition (6) is thus satisfied and the wellposedness of the residual equation follows. Global wellposedness of the original problem (1) is obtained by combining the wellposedness of (3) and that of heat equations.

## 6 An illustration of an example

The largest critical space known for the 3D Navier-Stokes equations is the Besov space  $B_{\infty,\infty}^{-1}$ . Bourgain and Pavlović [2] have shown that the Navier-Stokes system is illposed in the critical space. We shall show that the conditions set out for initial data in the main theorem allows arbitrary large  $B_{\infty,\infty}^{-1}$  norm. Chemin, Gallagher, Paicu and Zhang (e.g. [6]) have also provided different examples of arbitrarily large initial data in  $B_{\infty,\infty}^{-1}$  that the 3D Navier-Stokes system is globally wellposed under those data.

Let us first recall the theorem we are trying to establish:

**Theorem 4.** *For any  $M > 0$ , there exists initial data  $u_0^1, u_0^2, u_0^3$  satisfying that conditions in Theorem 1 such that each component  $u_0^i$  is large in the following sense:*

$$\|u_0^i\|_{B_{\infty,\infty}^{-1}} > M$$

**Remark 3.** Notice that by the nature of our initial condition being a two dimensional object, and from the embedding that in 2D,  $L^2 \hookrightarrow B_{\infty,\infty}^{-1}$ , the data demonstrated below will also have large energy.

*Proof.* Consider the function  $u_0^1$  with frequency support on  $0 \times [N, 2N] \times [N, 2N]$ , we can write it in terms of its Fourier expansion

$$u_0^1(x_2, x_3) = \sum_{m_2=N}^{2N} \sum_{m_3=N}^{2N} a_{m_2 m_3} e^{im_2 x_2} e^{im_3 x_3}$$

We shall pick the values  $a_{m_2 m_3}$  such that at time  $t = 1/N^2$ , they are all equal, i.e.

$$a_{m_2 m_3} e^{-\frac{m_2^2 + m_3^2}{N^2}} = A$$

for some  $A$ . We shall also require that the  $L^2$  norm of  $u_0^1$  to be equal to  $C^{-1} \log N$ , i.e.

$$\sum_{m_2=N}^{2N} \sum_{m_3=N}^{2N} a_{m_2 m_3}^2 = C^{-1} \log N$$

It is clear that we can find coefficients  $a_{m_2 m_3}$  that satisfy the above conditions.

For  $u_0^2$ , we pick a similar function, but localized in frequency on the support  $[N^{1+\epsilon}, 2N^{1+\epsilon}] \times 0 \times [N^{1+\epsilon}, 2N^{1+\epsilon}]$  instead. More precisely, if we write

$$u_0^2(x_1, x_3) = \sum_{m_1=N^{1+\epsilon}}^{2N^{1+\epsilon}} \sum_{m_3=N^{1+\epsilon}}^{2N^{1+\epsilon}} b_{m_1 m_3} e^{im_2 x_2} e^{im_3 x_3},$$

then the coefficients  $b_{m_1 m_3}$  satisfies the following:

$$b_{m_1 m_3} e^{-\frac{m_1^2 + m_3^2}{N^2(1+\epsilon)}} = B$$

for some  $B$  and

$$\sum_{m_1=N^{1+\epsilon}}^{2N^{1+\epsilon}} \sum_{m_3=N^{1+\epsilon}}^{2N^{1+\epsilon}} b_{m_1 m_3}^2 = C^{-1} \log N^{1+\epsilon}$$

Lastly, we shall take  $u_0^3 = (C^{-1} \delta \log N^{1+\epsilon})^{1/2} e^{ix_1} e^{ix_2}$ .

By construction, the data  $(u_0^1, u_0^2, u_0^3)$  satisfies the conditions of theorem (1), the only task remaining is to compute the  $B_{\infty, \infty}^{-1}$  norm of the data.

$$\begin{aligned} \|e^{\frac{\Delta}{N^2}} u_0^1\|_{L^\infty} &\geq \sum_{m_2=N}^{2N} \sum_{m_3=N}^{2N} a_{m_2 m_3} e^{-\frac{m_2^2 + m_3^2}{N^2}} \\ &= N \left( \sum_{m_2=N}^{2N} \sum_{m_3=N}^{2N} a_{m_2 m_3}^2 e^{-\frac{2(m_2^2 + m_3^2)}{N^2}} \right)^{1/2} \\ &\gtrsim N \left( \sum_{m_2=N}^{2N} \sum_{m_3=N}^{2N} a_{m_2 m_3}^2 \right)^{1/2} \\ &= N \|u_0^1\|_{L^2} \\ &= C^{-1/2} N (\log N)^{1/2} \end{aligned}$$

On the first equality above, we use the condition that at  $t = N^{-2}$ , the summands are all equal.

By definition,

$$\begin{aligned} \|u_0^1\|_{B_{\infty, \infty}^{-1}} &= \sup_{t>0} t^{1/2} \|e^{t\Delta} u_0^1\|_{L^\infty} \\ &\geq N^{-1} \|e^{\frac{\Delta}{N^2}} u_0^1\|_{L^\infty} \\ &\geq C^{-1/2} (\log N)^{1/2} \end{aligned}$$

Similarly, we can show that  $\|u_0^2\|_{B_{\infty,\infty}^{-1}} \geq C^{-1/2}(\log N^{1+\epsilon})^{1/2}$

Lastly, we can compute  $\|u_0^3\|_{B_{\infty,\infty}^{-1}} = (C^{-1}\delta \log N^{1+\epsilon})^{1/2}$ . By taking  $N$  to be large, we can guarantee that the  $B_{\infty,\infty}^{-1}$  norm of all three components to be larger than  $M$  for any  $M$

□

## 7 Concluding Remarks

By considering initial data with conditions in the frequency support, we can reduce and control the interaction between different components coming from the nonlinear term of the equation, and allows a perturbative approach as the equations are now closed to a decoupled set of heat equations. It will be interesting to consider a relaxation of such conditions, for example, when the initial data is the sum of two 2D data, say one in the  $x - y$  direction and the other in the  $y - z$  direction, with some amount of frequency separation. The reduction in this case would be to the established results of global wellposedness of 2D Navier-Stokes instead of the heat equations. Further considerations such as above will be addressed in a future article.

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